A subexponential upper bound for van der Waerden numbers $W(3,k)$

By

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Abstract

We show an improved upper estimate for van der Waerden number $W(3,k)$: there is an absolute constant $c>0$ such that if $\{1,\dots,N\}=X\cup Y$ is a partition such that X does not contain any arithmetic progression of length 3 and Y does not contain any arithmetic progression of length k then

$$N \leqslant \exp(O(k^{1-c}))\,.$$

1 Introduction

Let k and l be positive integers. The van der Waerden number $W(k,l)$ is the smallest positive integer N such that in any partition $\{1,\dots,N\}=X\cup Y$ there is an arithmetic progression of length k in X or an arithmetic progression of length l in Y. The existence of such numbers was established by van der Waerden [20], however the order of magnitude of $W(k,l)$ is unknown for $k,l\geqslant 3$. Clearly, $W(k,l)$ is related to Szemerédi's theorem on arithmetic progressions [18] and any effective estimate in this theorem leads to an upper bound on the van der Waerden numbers. Currently best known bounds in the most important diagonal case are

$$(1-o(1))\frac{2^{k-1}}{ek} \leqslant W(k,k) \leqslant 2^{2^{2^{2^{2^{k+9}}}}}\,.$$

The upper bound follows from the famous work of Gowers [11] and the lower bound was proved by Szabó [17] using probabilistic argument. Furthermore, Berlekamp [3] showed that if $k-1$ is a prime number then

$$W(k,k) \geqslant (k-1)2^{k-1}\,.$$

Another very intriguing instance the problem are numbers $W(3,k)$ as they are related to Roth's theorem [14] that provides more efficient estimates for sets avoiding three-terms arithmetic progressions. Let us denote by $r(N)$ the size of the largest progression-free subset of $\{1,\dots,N\}$. We know that

$$r(N) \ll \frac{N}{\log N^{1-o(1)}}\,, \tag{1}$$

see [4, 5, 15, 16]. However this bound is not strong enough to imply a subexponential estimate for $W(3,k)$.

Green [12] proposed a very clever argument based on arithmetic properties of sumsets to bound $W(3,k)$. Building on this method and applying results from [9] it was showed in [10] that

$$W(3,k) \leqslant \exp(O(k\log k))\,.$$

The best known lower bound was obtained by Li and Shu [13] (see also [7]), who showed that

$$W(3,k) \gg \Big(\frac{k}{\log k}\Big)^2.$$

The purpose of this paper is to prove a subexponential bound on $W(3,k)$.

Theorem 1 *There are absolute constants $C,c>0$ such that for every k we have*

$$W(3,k) \leqslant \exp(Ck^{1-c})\,.$$

Our argument is based on the method of [16], which explores in details the structure of a large spectrum. This method can be partly applied (see Lemma 5) in our approach and it deals only with a progression-free partition class. The second part of the proof exploits the structure of both partition classes and in this case the argument of [16] has to be significantly modified.

2 Notation

The Fourier coefficients of a function $f:\mathbb{Z}/N\mathbb{Z}\to\mathbb{C}$ are defined by

$$\widehat{f}(r)=\sum_{x=0}^{N-1} f(x)e^{-2\pi ixr/N},$$

where $r\in\mathbb{Z}/N\mathbb{Z}$. The inversion formula states that

$$f(x)=\frac{1}{N}\sum_{x=0}^{N-1}\widehat{f}(r)e^{2\pi ixr/N}.$$

We denote by $1_A(x)$ the indicator function of set A. Thus using the inversion formula and the fact that $(\widehat{1_A*1_B})(r)=\widehat{1_A}(r)\widehat{1_B}(r)$ one can express the number of three–term arithmetic progressions (including trivial ones) by

$$\frac{1}{N}\sum_{r=0}^{N-1}\widehat{1_A}(r)^2\widehat{1_A}(-2r)=|A|\,.$$

Parseval's identity asserts in particular that

$$\sum_{r=0}^{N-1}|\widehat{1_A}(r)|^2=|A|N\,.$$

Let $\theta \geqslant 0$ be a real number, the $\theta-$spectrum of A is defined by

$$\Delta_\theta(A) = \big\{r \in \mathbb{Z}/N\mathbb{Z} : |\widehat{1_A}(r)| \geqslant \theta|A|\big\}.$$

If A is specified then we write Δ_θ instead of $\Delta_\theta(A)$.

By the span of a finite set S we mean

$$\text{Span}\,(S) = \Big\{\sum_{s\in S}\varepsilon_s s : \varepsilon_s \in \{-1,0,1\} \text{ for all } s \in S\Big\}$$

and the dimension of A is defined by

$$\dim(A) = \min\big\{|S| : A \subseteq \text{Span}\,(S)\big\}\,.$$

Chang's Spectral Lemma provides an upper bound for the dimension of a spectrum.

Lemma 2 [8] *Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$ be a set of size $|A| = \delta N$ and let $\theta > 0$. Then*

$$\dim(\Delta_\theta(A)) \ll \theta^{-2}\log(1/\delta)\,.$$

We are going to use Bohr sets [6] to prove the main result. Let $\Gamma \subseteq \widehat{G}$ and $\gamma \in (0, \frac{1}{2}]$ then the Bohr set generated by Γ with radius γ is

$$B(\Gamma,\gamma) = \big\{x \in \mathbb{Z}/N\mathbb{Z} :\ \|tx/N\| \leqslant \gamma \text{ for all } t \in \Gamma\big\}\,,$$

where $\|x\| = \min_{y\in\mathbb{Z}}|x-y|$. The rank of B is the size of Γ and we denote it by $\text{rk}(B)$. Given $\eta > 0$ and a Bohr set $B = B(\Gamma,\gamma)$, by B_η we mean the Bohr set $B(\Gamma,\eta\gamma)$. We will use two basic properties of Bohr sets concerning its size and regularity, see [19].

Lemma 3 [6] *For every $\gamma \in (0, \frac{1}{2}]$ we have*

$$\gamma^{|\Gamma|}N \leqslant |B(\Gamma,\gamma)| \leqslant 8^{|\Gamma|+1}|B(\Gamma,\gamma/2)|\,.$$

We call a Bohr set $B(\Gamma,\gamma)$ regular if for every η, where $|\eta| \leqslant 1/(100|\Gamma|)$ we have

$$(1-100|\Gamma||\eta|)|B| \leqslant |B_{1+\eta}| \leqslant (1+100|\Gamma||\eta|)|B|.$$

Bourgain [6] showed that regular Bohr sets are ubiquitous.

Lemma 4 [6] *For every Bohr set $B(\Gamma,\gamma)$, there exists γ' such that $\frac{1}{2}\gamma \leqslant \gamma' \leqslant \gamma$ and $B(\Gamma,\gamma')$ is regular.*

3 Proof of Theorem 1

Our main tool is the next lemma, which can be extracted from [16] (see Lemmas 7, 9, 12 and 13). Its proof makes use of the deep result by Bateman and Katz in [1, 2] describing the structure of the large spectrum.

Lemma 5 [16] *There exists an absolute constant $c > 0$ such that the following holds. Let $A \subseteq \mathbb{Z}/N\mathbb{Z}$, $|A| = \delta N$ be a set without any non-trivial arithmetic progressions of length three and such that*

$$\sum_{r\colon \delta^{1+c}|A| \leqslant |\widehat{1_A}(r)| \leqslant \delta^{1/10}|A|} |\widehat{1_A}(r)|^3 \geqslant \frac{1}{10}\delta^{c/5}|A|^3 . \qquad (2)$$

Then there is a regular Bohr set B with $\mathrm{rk}(B) \ll \delta^{-1+c}$ *and radius* $\Omega(\delta^{1-c})$ *such that for some t*

$$|(A+t) \cap B| \gg \delta^{1-c}|B|.$$

Furthermore, we apply Bloom's iterative lemma, that provides a density increment by a constant factor greater than 1 for progression-free sets and Sanders' lemma on a containment of long arithmetic progressions in dense subsets of regular Bohr sets.

Lemma 6 [4] *There exists an absolute constant $c_1 > 0$ such that the following holds. Let $B \subseteq \mathbb{Z}/N\mathbb{Z}$ be a regular Bohr set of rank d. Let $A_1 \subseteq B$ and $A_2 \subseteq B_\varepsilon$, each with relative densities α_i. Let $\alpha = \min(c_1, \alpha_1, \alpha_2)$ and assume that $d \leqslant \exp(c_1(\log^2(1/\alpha))$. Suppose that B_ε is also regular and $c_1\alpha/(4d) \leqslant \varepsilon \leqslant c_1\alpha/d$. Then either*

(i) there is a regular Bohr set B' of rank $\mathrm{rk}(B') \leqslant d + O(\alpha^{-1}\log(1/\alpha))$ *and size*

$$|B'| \geqslant \exp\big(-O(\log^2(1/\alpha)(d + \alpha^{-1}\log(1/\alpha)))\big)|B|$$

such that

$$|(A_1 + t) \cap B'| \gg (1 + c_1)\alpha_1|B'|$$

for some $t \in \mathbb{Z}/N\mathbb{Z}$;

(ii) or there are $\Omega(\alpha_1^2\alpha_2|B||B_\varepsilon|)$ three-term arithmetic progressions $x + y = 2z$ with $x, y \in A_1, z \in A_2$;

Lemma 7 [15] *Let $B(\Gamma, \gamma) \subseteq \mathbb{Z}/N\mathbb{Z}$ be a regular Bohr set of rank d and let ε be a positive number satisfying $\varepsilon^{-1} \ll \gamma d^{-1}N^{1/d}$. Suppose that $A \subseteq B$ contains at least a proportion $1 - \varepsilon$ of $B(\Gamma, \gamma)$. Then A contains an arithmetic progression of length at least $1/(4\varepsilon)$.*

Proof of Theorem 1. Put $M = W(3,k) - 1$ and let $\{1, \ldots, M\} = X \cup Y$ be a partition such that X and Y avoid 3 and k-term arithmetic progressions respectively. Clearly, we may assume that $M \geqslant 100k$ hence

$$|Y| \leqslant M - \lfloor M/k \rfloor \leqslant M - M/(2k),$$

as no block of k consecutive numbers is contained in Y and therefore $|X| \geqslant M/(2k)$. Let N be any prime number satisfying $2M < N \leqslant 4M$. We embed $\{1, \ldots, M\} = X \cup Y$ in $\mathbb{Z}/N\mathbb{Z}$ in a natural way and observe that X and Y possess the same properties. Put $|X|=\delta N$. First let us assume that

$$\sum_{r:\,\delta^{1+\mu}|X|\leqslant|\widehat{1_X}(r)|\leqslant\delta^{1/10}|X|}|\widehat{1_X}(r)|^3\geqslant\delta^{\mu/5}|X|^3\,, \tag{3}$$

Then by Lemma 5 there is $t\in\mathbb{Z}/N\mathbb{Z}$ and a regular Bohr set B^0 with $\mathrm{rk}(B^0)\ll\delta^{-1+c}$ and radius $\Omega(\delta^{1-c})$ such that

$$|(X+t)\cap B^0|\gg\delta^{1-c}|B|$$

for some absolute constant $c>0$. Writing $A_0=(X+t)\cap B_0$ we have

$$|X_0\cap B^0|\gg\alpha|B^0|\,,$$

where

$$\alpha\gg\delta^{1-c}\,.$$

By Lemma 3 we have

$$|B^0|\geqslant\exp\big(-O(\delta^{-1+c}\log(1/\delta))\big)N\,.$$

Next, we iteratively apply Lemma 6. Since after each step the density increases by factor $1+c_1$ it follows that after $l\ll\log(1/\alpha)$ steps case (ii) of Lemma 6 holds. Let B^i be Bohr sets obtained in the iterative procedure and observe that $\mathrm{rk}(B^i)\ll\alpha^{-1}\log^2(1/\alpha)$ for every $i\leqslant l$. Therefore, there are

$$\Omega(\alpha^3|B^k||B^l_\varepsilon|)$$

three-term arithmetic progressions in X, where $\varepsilon\geqslant c_1\alpha/(4\mathrm{rk}(B^l))\gg\alpha^2\log^2(1/\alpha)$. By Lemma 6 and Lemma 3 we have

$$|B^l|\geqslant\exp\big(-O(\alpha^{-1}\log^4(1/\alpha))\big)N\geqslant\exp\big(-O(\delta^{-1+c}\log^4(1/\delta))\big)N\,,$$

and

$$\begin{aligned}|B^l_\varepsilon| &\geqslant \exp\big(-O(\alpha^{-1}\log^3(1/\alpha))\big)\exp\big(-O(\alpha^{-1}\log^4(1/\alpha))\big)N\\ &\geqslant \exp\big(-O(\delta^{-1+c}\log^4(1/\delta))\big)N\,.\end{aligned}$$

Thus, X contains

$$\Omega(\delta^{3-3c}\exp\big(-O(\delta^{-1+c}\log^4(1/\delta))\big)N^2)$$

arithmetic progressions of length three. Since there are only $|X|$ trivial progressions in X it follows that

$$|X|\gg\delta^{3-3c}\exp\big(-O(\delta^{-1+c}\log^4(1/\delta))\big)N^2\,,$$

so

$$W(3,k)\ll N\ll\exp\big(O(\delta^{-1+c}\log^4(1/\delta)\big)\leqslant\exp\big(O(k^{1-c}\log^4 k)\big)\,.$$

Next let us assume that (3) does not hold. By Chang's lemma

$$d:=\dim(\Delta_{\delta^{1/10}}(X))\ll\delta^{-1/5}\log(1/\delta)$$

hence there is a set Λ such that $|\Lambda| = d$ and $\Delta_{\delta^{1/10}} \subseteq \mathrm{Span}\,(\Lambda)$. By Lemma 4 there is a regular Bohr set $B = B(\Lambda, \gamma)$ with radius $\gamma \gg \delta^3$. Let $\beta = \frac{1}{|B|}1_B$ then for every $r \in \Delta_{\delta^{1/10}}$ we have

$$\big|\widehat{\beta}(r) - 1\big| \leqslant \frac{1}{|B|}\sum_{b\in B}|e^{-2\pi i\lambda b/N} - 1| \leqslant \frac{2\pi}{|B|}\sum_{b\in B}\sum_{\lambda\in\Lambda}\|rb/N\| \leqslant 2\pi\delta^2\,, \tag{4}$$

and similarly $|\widehat{\beta}(2r) - 1| \ll \delta^2$. For $t \in \mathbb{Z}/N\mathbb{Z}$ put

$$f(t) = \beta * 1_X(t)$$

and note that if for some $t \in [4\gamma M, (1-4\gamma)M]$ we have $f(t) = \frac{1}{|B|}|X \cap (B+t)| \leqslant \delta^{1+c'}$, where $c' = c/20$, then since $B + t \subseteq [1, M]$ it follows that

$$|Y \cap (B+t)| \geqslant 1 - \delta^{1+c'}\,.$$

Therefore, by Lemma 7 either $\delta^{-1-c'} \gg \gamma d^{-1}N^{1/d}$ or Y contains an arithmetic progression of length $\frac{1}{4}\delta^{-1-c'}$. The former inequality implies that

$$k^{1+c'} \gg \gamma d^{-1}N^{1/d} \gg \delta^4 N^{O(\delta^{1/5}\log^{-1}(1/\delta))} \gg k^{-4}N^{O(k^{-1/5}\log^{-1}k)}\,,$$

so

$$W(3,k) \ll N \ll \exp\big(O(k^{1/5}\log^2 k)\big)\,.$$

If the second alternative holds then

$$\frac{1}{4}\delta^{-1-c'} < k$$

hence by (1)

$$k^{-1/(1+c')} \ll \delta \ll (\log N)^{-1+o(1)}$$

so

$$W(3,k) \leqslant N \ll \exp\big(O(k^{\frac{1}{1+c'}+o(1)})\big)\,.$$

Finally we can assume that for every $t \in [4\gamma M, (1-4\gamma)M]$ we have $f(t) \geqslant \delta^{1+c'}$. Let $T(X)$ denote the number of three-term arithmetic progression in X and let

$$T(f) = \sum_{x+y=2z} f(x)f(y)f(z)\,.$$

Then clearly

$$T(f) \gg \delta^{3+3c'}M^2 \gg \delta^{3+c/6}N^2 \tag{5}$$

and we will show that $T(X)$ does not differ much from $T(f)$

$$\begin{aligned}
\big|T(X) - T(f)\big| &= \frac{1}{N}\Big|\sum_{r=0}^{N-1}\widehat{1_X}(r)^2\widehat{1_X}(-2r) - \sum_{r=0}^{N-1}\widehat{f}(r)^2\widehat{f}(-2r)\Big| \\
&\leqslant \frac{1}{N}\sum_{r=0}^{N-1}|\widehat{1_X}(r)^2\widehat{1_X}(-2r)(1-\widehat{\beta}(r)^2\widehat{\beta}(-2r))| \qquad (6)\\
&= S_1 + S_2 + S_3,
\end{aligned}$$

where S_1, S_2 and S_3 summations of (6) respectively over $\Delta_{\delta^{1/10}}, \Delta_{\delta^{1+c}} \setminus \Delta_{\delta^{1/10}}$ and $\mathbb{Z}/N\mathbb{Z} \setminus \Delta_{\delta^{1+c}}$. By (4), (3), Parseval's formula and Hölder's inequality we have

$$S_1 \ll \delta^2 \frac{1}{N} \sum_{r \in \Delta_{\delta^{1/10}}} |\widehat{1_X}(r)|^3 \leqslant \delta^3 \sum_{r=0}^{N-1} |\widehat{1_X}(r)|^2 = \delta^2 |X|^2 ,$$

$$S_2 \leqslant \delta^{1+c/5} |X|^2$$

and

$$S_3 \leqslant \frac{2}{N} \delta^{1+c} |X| \sum_{r=0}^{N-1} |\widehat{1_X}(r)|^2 = \delta^{1+c} |X|^2 .$$

Thus,

$$\big|T(X) - T(f)\big| \ll \delta^{3+c/5} N^2,$$

so by (5) and the fact that X avoids non-trivial three-term arithmetic progression we have

$$|X| = T(X) \gg \delta^{3+c/6} N^2 ,$$

hence

$$W(3,k) \leqslant N \ll \delta^{-2-c/6} \ll k^3$$

which concludes the proof. □

References

[1] M. Bateman, N. Katz, *New bounds on cap sets,* Journal of AMS 2 (2012), 585–613.

[2] M. Bateman, N. Katz, *Structure in additively nonsmoothing sets,* arXiv:1104.2862v1.

[3] E. Berlekamp, *A construction for partitions which avoid long arithmetic progressions,* Canad. Math. Bull. 11 (1968) 409–414.

[4] T. Bloom, *A quantitative improvement for Roth's theorem on arithmetic progressions,* J. Lond. Math. Soc. 93 (2016), 643–663.

[5] T. Bloom, O. Sisask, *Logarithmic bounds for Roth's theorem via almost-periodicity,* Discrete Analysis 4 (2019), 20 pp.

[6] J. Bourgain, *On triples in arithmetic progression,* Geom. Funct. Anal. 9 (1999), 968–984.

[7] T. Brown, B. Landman, A. Robertson, *Bounds on some van der Waerden numbers,* J. Combin. Theory Ser. A 115 (2008), 1304–1309.

[8] M.–C. Chang, *A polynomial bound in Freiman's theorem,* Duke Math. J. 3 (2002), 399–419.

[9] E. Croot, I. Ruzsa, T. Schoen, *Arithmetic progressions in sparse sumsets,* Combinatorial number theory, de Gruyter, Berlin, 2007, 157–164.

[10] K. Cwalina, T. Schoen, *Tight bounds on additive Ramsey-type numbers,* J. Lond. Math. Soc. 96 (2017), 601–620.

[11] W. T. Gowers, *A new proof of Szemerédi's theorem,* Geom. Funct. Anal. 11 (2001), 465–588.

[12] B. Green, *Arithmetic progressions in sumsets,* Geom. Funct. Anal. 12 (2002), 584–597.

[13] Y. Li, J. Shu, *A lower bound for off-diagonal van der Waerden numbers,* Adv. in Appl. Math. 44 (2010), 243–247.

[14] K. F. Roth, *On certain sets of integers,* J. London Math. Soc. 28 (1953), 104–109.

[15] T. Sanders, *On Roth's Theorem on Progressions,* Ann. of Math. 174 (2011), 619–636.

[16] T. Schoen, *Improved bound in Roth's theorem on arithmetic progressions,* arXiv:2005.01145.

[17] Z. Szabó, *An application of Lovász' local lemma–A new lower bound for the van der Waerden numbers,* Random Structures Algorithms 1 (1990), 343–360.

[18] E. Szemerédi, *On sets of integers containing no k elements in arithmetic progression,* Acta Arith. 27 (1975), 199–245.

[19] T. Tao, V. Vu, *Additive combinatorics,* Cambridge University Press 2006.

[20] B. L. van der Waerden, *Beweis einer Baudetschen Vermutung,* Nieuw Arch. Wiskd. 15 (1927), 257–271.

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