Near optimal bounds in Freiman's theorem

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Abstract

We prove that if for a finite set A of integers we have $|A+A| \leq K|A|$, then A is contained in a generalized arithmetic progression of dimension at most $K^{1+C(\log K)^{-1/2}}$ and size at most $\exp(K^{1+C(\log K)^{-1/2}})|A|$ for some absolute constant C. We also discuss a number of applications of this result.

Introduction

Freiman's theorem [8] is one of the most fundamental theorems in additive number theory. It asserts that if for a finite $A \subseteq \mathbb{Z}$ we have $|A+A| \leq K|A|$, then A is a subset of a d dimensional generalized arithmetic progression $P_1 + \cdots + P_d$ with $|P_1| \cdots |P_d| = |A|f$, where d and f depend only on K (here P_i stand for usual arithmetic progressions). This result provides a complete description of sets for which the sumset is not much larger than the set itself. Freiman's theorem have a plethora of deep applications to many important problems as the recent one from the papers of Gowers [12], [13] who used it to get an effective version of Szemerédi's theorem (for further papers which make an important use of Freiman's result see for example [4], [6],[26], [27]–[29]). However, most of applications require a quantitative version of Freiman's theorem. It is easy to see that one cannot do better than $d(K) \leq K-1$ and $f(K) = e^{O(K)}$. The estimates which followed from Freiman's original proof were quite poor. The first useful estimates for d and f were provided by Ruzsa [20], whose ingenious argument combines techniques from different fields. Ruzsa's bounds were good enough for many applications and his approach paved the way for further improvements. The next substantial progress was made by Chang [3], who showed that one can take $d \leq K^{2+o(1)}$ and $f(K) \leq e^{K^{2+o(1)}}$. Her proof coupled Ruzsa's main ideas with the so-called Chang's Spectral Lemma and Chang's Covering Lemma. Further improvements were due to Sanders [23] who used a much expanded version of Bogolyubov-Ruzsa's lemma to obtain the best currently known bounds $d(K) \leq K^{7/4+o(1)}$ and $f(K) \leq e^{K^{7/4+o(1)}}$. The main result of this note is the following theorem which gives nearly optimal estimates for d and f.

Theorem 1 *There exists an absolute constant C such that each set $A \subseteq \mathbb{Z}$ satisfying $|A+A| \leq K|A|$ is contained in a generalized arithmetic progression of dimension at most $d(K)$ and size at most $f(K)|A|$ with $d(K) \leq K^{1+C(\log K)^{-1/2}}$ and $f(K) \leq \exp(K^{1+C(\log K)^{-1/2}})$.*

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Our approach is yet another follow up of Ruzsa's original argument and our improvement relies on further refinement of Bogolyubov-Ruzsa's lemma. A new addition to this set up is an elementary result, Lemma 3, which roughly says that one can find two sets 'closely related' to A, which have much better additive properties. As a consequence of our method we infer that there is a proper arithmetic progression P of dimension $O_\varepsilon(K^\varepsilon)$ and size bounded from above by $K^{O(1)}|A|$ such that $|A\cap P| \ge \exp(-O_\varepsilon(K^\varepsilon))|A|$ (for a precise statement see Theorem 8 in the last section). This fact has been already anticipated by Chang [3], Gowers [14] and Green and Tao [11] (Conjecture 1.6) and is often required in applications. Remarkably, Green and Tao [11] proved that it is equivalent to a inverse theorem for Gowers U^3-norm. It can be also considered as a first step towards resolving Freiman-Ruzsa polynomial conjecture. We also remark that, using Freiman-Bilu [1] argument (see also [3]), one cannot deduce from Theorem 1 that A is contained in a progression with linear dimension and size at most $\exp(K^{1+C(\log K)^{-1/2}})|A|$. The reason is that we cannot guarantee our progression to be proper. Our proof can be modified to achieve it as well but in this case the size of the progression goes up to $\exp(K^{2+o(1)})|A|$.

Finally, our method can be directly applied to Green-Ruzsa's [10] proof of Freiman's theorem in the 'torsion' case, giving the following statement, which proof we omit here (further improvement of this theorem will appear in [7]).

Theorem 2 *Let G be an arbitrary abelian group and let $A \subseteq G$ satisfy $|A+A| \le K|A|$. Then A is contained in a coset progression $P+H$ of dimension $d(K) \le (K+2)^{3+C(\log(K+2))^{-1/2}}$ and size $f(K)|A| \le \exp((K+2)^{3+C(\log(K+2))^{-1/2}})|A|$ for some absolute constant C.*

The paper is organized as follows. The next section contains some basic definitions and facts we use in the note. The main part of the paper is Section 3, where we prove, key for our argument, Lemma 3. In order to keep the paper self-contained, we briefly outline the rest of Ruzsa's argument (with Chang's refinements) in Section 4. We conclude the note with some applications of our results and a few comments on our approach and its possible further developments.

Preliminaries

This part of the note contains basic definitions and notation we shall use later on.

As usual, we set

$$A+B = \{a+b : a\in A, b\in B\},$$

and the k-fold sumset of A is denoted by kA. By a generalized arithmetic progression of dimension d we mean every set of the form $P = P_1+\dots+P_d$, where $P_1,\dots,P_d$ are usual arithmetic progressions. The size of P is defined as the product $|P_1|\cdots|P_d|$. The dimension and the size of progression P are denoted by $\dim(P)$ and $\mathrm{size}(P)$, respectively. If each $x\in P$ has unique representation $x = p_1+\dots+p_d,\ p_i\in P_i$, we say that P is proper. Then the cardinality of P is equal to its size.

Let G, H, be abelian groups and $A\subseteq G$, $B\subseteq H$. We say that A is F_k-isomorphic (i.e., Freiman isomorphic of order k) to B if there exists a bijective map $\varphi : A\to B$ such that

$$x_1+\dots+x_k = y_1+\dots+y_k$$

if and only if

$$\varphi(x_1) + \cdots + \varphi(x_k) = \varphi(y_1) + \cdots + \varphi(y_k)$$

for every $x_1, \ldots, x_k, y_1, \ldots, y_k \in A$.

We call (x, x', y, y') an additive quadruple if $x + y = x' + y'$. The number of additive quadruples in $X^2 \times Y^2$ is denoted be $\mathsf{E}(X, Y)$.

In this paper by $\mathbb{Z}_n$ we always mean $\mathbb{Z}/n\mathbb{Z}$. The Fourier coefficients of the indicator function of a set $X \subseteq \mathbb{Z}_n$ are defined by

$$\hat{X}(s) = \sum_{x \in X} e^{-2\pi i x s/n},$$

where $s \in \mathbb{Z}_n$. Parseval's formula states that $\sum_{s=0}^{n-1} |\hat{X}(s)|^2 = |X|n$. We observe also that for $X, Y \subseteq \mathbb{Z}_n$ we have

$$\mathsf{E}(X, Y) = \frac{1}{n} \sum_{s=0}^{n-1} |\hat{X}(s)|^2 |\hat{Y}(s)|^2. \tag{1}$$

Our argument makes use of Plünnecke-Ruzsa inequality, which says, in particular, that if $|A + A| \le K|A|$ then for all $k, l \in \mathbb{N}$ we have

$$|kA - lA| \le K^{k+l}|A|.$$

By $\|\alpha\|$ we denote the distance from a to the nearest integer. For $\Gamma \subseteq \mathbb{Z}_n$ and $\gamma > 0$ the Bohr set $B(\Gamma, \gamma)$ is defined as

$$B(\Gamma, \gamma) = \big\{x \in \mathbb{Z}_n : \|gx/n\| \le \gamma \text{ for every } g \in \Gamma\big\}.$$

The Main Lemma

The key idea of our approach is to identify a relatively large subset of A, which has better additive properties than A. One of possible ways would be to find $A' \subseteq A$ such that $|kA'| \le K^{\varepsilon k}|A'|$ for not too big k. A result of this kind was proved in [17], but under a stronger assumption on A and, unfortunately, this method does not seem to work in our general setting. Let us remark however that Bogolyubov-Ruzsa's lemma uses the number of additive quadruples rather than the size of the subset. This suggests the following approach: using the assumption $|A+A| \le K|A|$ one should find for some small k a relatively dense sets $X, Y \subseteq kA - kA$, with $\mathsf{E}(X, Y)$ much bigger than one can expect from the size of $A + A$. As far as we know, the first such result was proved by Katz and Koester [15]. They showed that there is $B \subseteq A \pm A$ with $\mathsf{E}(B, B) \gg |B|^3/K^{36/37}$. Lemma 3 provides a much better estimate for $\mathsf{E}(X, Y)$ in a 'highly non-symmetric' case, when one set is large while the other is small (a similar result for a diagonal case would give much better result – see our concluding remarks below). Finally, let us also mention that Sanders in recent works [22] (see also [25] for related theorems), [24] used a similar result for a different purpose – that of improving the bounds in many summed versions of Roth-Meshulam theorem.

Lemma 3 *Suppose that A is a subset of an abelian group and $|A + A| \le K|A|$. Then for every $\varepsilon > 0$ there are sets $X \subseteq A$ and $Y \subseteq A + A$ such that $|X| \ge (2K^2)^{-2^{1/\varepsilon}}|A|$, $|Y| \ge |A|$ and $\mathsf{E}(X, Y) \ge K^{-2\varepsilon}|X|^2|Y|$.*

Proof. For a set $B \subseteq A$ denote by $D = D(B)$ the set of all $t \in B - B$, which has at least

$$\frac{|B|^2}{2|B-B|} \geq \frac{|B|^2}{2|A-A|} \geq \frac{|B|^2}{2K^2|A|}$$

representations as $b-b',\ b,b' \in B$ (the last inequality follows from Plünnecke-Ruzsa's inequality). Note that $|D| \geq |B|/2$.

We first show that there exists a set $B \subseteq A$ such that $|B| \geq (2K^2)^{-2^{1/\varepsilon}+1}|A|$ and

$$|A + B_t| \geq K^{-\varepsilon}|A+B| \tag{2}$$

for all $t \in D = D(B)$, where $B_t = B \cap (B+t)$. We shall construct B in the following iterative procedure. We start with $A^1 = A$. Now suppose that A^l with $|A^l| \geq (2K^2)^{-2^{l-1}+1}|A|$ has already been defined. If there exists $t_0 \in D(A^l)$ with $|A + A^l_{t_0}| < K^{-\varepsilon}|A + A^l|$, then we put $A^{l+1} = A^l_{t_0}$. Observe that

$$|A^{l+1}| \geq \frac{|A^l|^2}{2|A^l - A^l|} \geq \frac{|A^l|^2}{2|A-A|} \geq (2K^2)^{-2^l+1}|A|.$$

If one cannot find such a t_0, then (2) is satisfied with $B = A^l$ and we are done.

Note that after k iterates we obtain a set A^{k+1} such that $|A^{k+1}| \geq (2K^2)^{-2^k+1}|A|$ and

$$|A + A^{k+1}| < K^{-\varepsilon k}|A+A| = K^{1-\varepsilon k}|A|.$$

But $|A + A^{k+1}| \geq |A|$, so that the procedure must terminate after at most $\lceil 1/\varepsilon \rceil - 1$ steps.

Let B be a set satisfying (2). Observe that for every $t \in D$ we have

$$A + B_t \subseteq A + B \qquad \text{and} \qquad A + B_t \subseteq A + B + t,$$

so that

$$|(A+B) \cap (A+B+t)| \geq |A + B_t| \geq K^{-\varepsilon}|A+B|. \tag{3}$$

Furthermore, there are at least $|B|^2/2$ pairs $(b,b') \in B^2$ with $b - b' \in D$. Therefore, by an averaging argument we find $b_0 \in B$ such that

$$|(B - b_0) \cap D| \geq |B|/2.$$

We put $D' = B \cap (D + b_0) \subseteq A$. Applying Cauchy-Schwarz inequality we infer that

$$\begin{aligned}
\mathsf{E}(D', A+B) &= \sum_{t,t' \in D'} |(A+B+t) \cap (A+B+t')| \\
&\geq \sum_{t,t' \in D'} |(A+B+t) \cap (A+B+t') \cap (A+B+b_0)| \\
&= \sum_{x \in A+B+b_0} |\{t \in D' : x \in A+B+t\}|^2 \\
&\geq |A+B|^{-1} \Big(\sum_{x \in A+B} |\{t \in D' : x \in A+B+t\}| \Big)^2 \\
&= |A+B|^{-1} \Big(\sum_{t \in D'} |(A+B+t) \cap (A+B+b_0)| \Big)^2 \\
&\geq K^{-2\varepsilon}|D'|^2|A+B|.
\end{aligned}$$

Since $|D'| \geq |B|/2 \geq (2K^2)^{-2^{1/\varepsilon}}|A|$, the assertion follows for $X = D'$ and $Y = A + B$. □

Proof of Theorem 1

In order to show Theorem 1 one needs to mimic Ruzsa-Chang's proof using the above Lemma 3. In order to make the note self-contained we briefly sketch their argument.

Let n be a prime satisfying

$$48|12A - 12A| > n > 24|12A - 12A|.$$

By Plünnecke-Ruzsa's inequality it follows that $n \leq 48K^{24}|A|$. By Ruzsa's lemma (see Theorem 2 in [19]) there exists a set $A' \subseteq A$ such that $|A'| \geq |A|/12$, which is F_{12}-isomorphic to $T \subseteq \mathbb{Z}_n$. Clearly, $|T+T| \leq 12K|T|$. Let $X \subseteq T$ and $Y \subseteq T+T$ be sets such that the assertion of Lemma 3 holds with

$$\varepsilon = (\log K)^{-1/2}$$

Thus, we have

$$|X| \geq (2(12K)^2)^{-2^{1/\varepsilon}}|T| \geq 2^{-10}(288K^2)^{-2^{1/\varepsilon}}K^{-24}n.$$

The next ingredient of our approach is Chang's Spectral Lemma [3] (see also [10]). For $\Gamma \subseteq \mathbb{Z}_n$ define

$$\mathrm{Span}(\Gamma) = \Big\{\sum_{g\in\Gamma} \varepsilon_g g : \varepsilon_g \in \{-1, 0, 1\}\Big\}.$$

Then the spectral lemma can be stated as follows.

Lemma 4 *Let $X \subseteq \mathbb{Z}_n$ and $\Lambda = \{r \in \mathbb{Z}_n : |\hat{X}(r)| \geq \lambda|X|\}$. Then there is a set $\Gamma \subseteq \mathbb{Z}_n$ such that $|\Gamma| \leq 2\lambda^{-2}\log(n/|X|)$ and $\Lambda \subseteq \mathrm{Span}(\Gamma)$.*

Let us emphasize at this point that the above Chang's Spectral Lemma is absolutely crucial for our argument. While in the Chang's paper one can omit it and still get reasonable bounds $d(K) \leq K^{O(1)}$ and $f(K) \leq e^{K^{O(1)}}$, without Lemma 4 our approach completely fails. In fact, as it is easy to see, Chang's Spectral Lemma is especially useful if λ is much bigger than the density of X, and we apply it precisely in such a case using its full potential.

Our goal is to get a version of Bogolyubov-Ruzsa's lemma for sets X and Y.

Lemma 5 *The set $X + Y - X - Y$ contains a Bohr set $B(\Gamma, \gamma)$ such that $|\Gamma| \ll K^{3\varepsilon}\log K$ and $\gamma \gg K^{-3\varepsilon}\log^{-1} K$.*

P r o o f. Define

$$\Lambda = \big\{r \in \mathbb{Z}_n : |\hat{X}(r)| \geq K^{-\varepsilon}|X|/2\big\}.$$

We shall show that $B = B(\Lambda, 1/6) \subseteq X+Y-X-Y$. Let $r(x)$ be the number of representations of x in $X+Y-X-Y$. If $x \in B$, then by (1) and Parseval's formula we have

$$\begin{aligned}
r(x) &= \frac{1}{n}\sum_{s=0}^{n-1} |\hat{X}(s)|^2|\hat{Y}(s)|^2 e^{2\pi i x s/n} \\
&\geq \frac{1}{n}\sum_{s\in\Lambda} |\hat{X}(s)|^2|\hat{Y}(s)|^2 \cos(2\pi x s/n) - \frac{1}{n}\sum_{s\notin\Lambda} |\hat{X}(s)|^2|\hat{Y}(s)|^2 \\
&\geq \frac{1}{2n}\sum_{s\in\Lambda} |\hat{X}(s)|^2|\hat{Y}(s)|^2 - \frac{1}{n}\sum_{s\notin\Lambda} |\hat{X}(s)|^2|\hat{Y}(s)|^2 \\
&\geq \frac{1}{2n}\sum_{s=0}^{n-1} |\hat{X}(s)|^2|\hat{Y}(s)|^2 - \frac{3}{2n}\sum_{s\notin\Lambda} |\hat{X}(s)|^2|\hat{Y}(s)|^2 \\
&> \mathsf{E}(X,Y)/2 - \frac{3}{8n}K^{-2\varepsilon}|X|^2\sum_{s=0}^{n-1}|\hat{Y}(s)|^2 \\
&\geq \mathsf{E}(X,Y)/2 - (3/8)K^{-2\varepsilon}|X|^2|Y| > 0.
\end{aligned}$$

By Chang's Spectral Lemma 4, there is a set Γ such that $|\Gamma| \ll K^{2\varepsilon}\log(n/|X|) \ll K^{3\varepsilon}\log K$ and $\Lambda \subseteq \mathrm{Span}(\Gamma)$, so that

$$B(\Gamma, 1/(6|\Gamma|)) \subseteq B(\Lambda, 1/6) \subseteq X+Y-X-Y. \quad \square$$

Remark We proved Bogolyubov-Ruzsa's lemma in a standard way, but one can easily strengthen the assertion by removing one summand. Indeed, by (3) we have

$$\frac{1}{n}\sum_{s=0}^{n-1} \hat{X}(s)|\hat{Y}(s)|^2 e^{-2\pi i b_0 s/n} \geq K^{-\varepsilon}|X||Y|.$$

Therefore, using similar argument as in the proof of Lemma 5, we infer that a shift of $B(\Gamma,\gamma)$ is contained in $X+Y-Y$.

Let us first recall Ruzsa's lemma on geometry of numbers.

Lemma 6 *Let $B(\Gamma,\gamma)$ be a Bohr set in $\mathbb{Z}_n$. Then there is a proper progression $P \subseteq B(\Gamma,\gamma)$ of dimension $|\Gamma|$ and size at least $(\gamma/|\Gamma|)^{|\Gamma|}n$.*

Thus, by Lemma 1, one can find a proper progression $P \subseteq X+Y-X-Y \subseteq 3T-3T$ with $\dim(P) \leq |\Gamma| \ll K^{3\varepsilon}\log K$ and

$$\mathrm{size}(P) \geq \exp(-K^{3\varepsilon}(\log K)^{3/2+o(1)})n.$$

Let P' be the image of P under considered F_{12}-isomorphism between A' and T. It induces a F_2-isomorphism between P and P', which implies that P' is also proper. For the last step of the argument we need the following Chang's Covering Lemma [3].

Lemma 7 *If $|A+A| \le K|A|$ and $|B+A| \le L|B|$ then there are sets $S_1, \dots, S_k$ each of size $\le 2K$ such that $A \subseteq B - B + (S_1 - S_1) + \cdots + (S_k - S_k)$ and $k \le 1 + \log(KL)$.*

Now, we have

$$|P' + A| \le |5A - 4A| \le K^9 |A| \le K^9 \exp(K^{3\varepsilon} (\log K)^{3/2+o(1)}) |P'|,$$

so that

$$A \subseteq P' - P' + (S_1 - S_1) + \cdots + (S_k - S_k)$$

with $k \le K^{3\varepsilon} (\log K)^{5/2+o(1)}$. Finally, A is contained in a progression Q with

$$\dim(Q) \le \dim(P) + 2kK \le K^{1+C(\log K)^{-1/2}}$$

and

$$\mathrm{size}(Q) \le 2^{\dim(P')} \mathrm{size}(P') 3^{2kK} \le e^{K^{1+C(\log K)^{-1/2}}} |A|$$

for some constant C. This completes the proof of Theorem 1.

Some Applications

We focus on consequences of a new version of Bogolyubov-Ruzsa lemma in Freiman's theorem, as one of the most important applications. However, as an immediate consequence of results proved in the previous section we obtain the following step in the direction of polynomial Freiman-Ruzsa conjecture, which may have also other applications.

Theorem 8 *There exists an absolute constant C such that if $A \subseteq \mathbb{Z}$ satisfies $|A+A| \le K|A|$, then there is a proper generalized arithmetic progression P with $\dim(P) \le K^{C(\log K)^{-1/2}}$ and $\mathrm{size}(P) \le 48K^{24}|A|$ such that $|A \cap P| \ge \exp(K^{C(\log K)^{-1/2}})|A|$.*

Using Theorem 8 and Theorem 1 one can directly improve many results based on Freiman's theorem. Here we mention only some of them. The most exciting case is Gowers theorem, however here the effect is not so impressive. It is proved in [13] that for every $k \in \mathbb{N}$ the density of a subset of $\{1, \dots, n\}$, which does not contain any k-term arithmetic progression does not exceed

$$(\log \log n)^{-c_k},$$

where $c_k = 2^{-2^{k+9}}$. Applying our version of Bogolyubov-Ruzsa's lemma one can at most increase the value of c_k. On the other hand, one can deduce from Lemma 5 (and in view of the Remark) that for each set $A \subseteq \{1, \dots, n\}$ of density $\delta = |A|/n$, one can find an arithmetic progression of length

$$c_\delta n^{e^{-c(\log 1/\delta)^{1/2}}}$$

in $5A$, for some absolute constant $c > 0$, beating the the previous best bound $c_\delta n^{c\delta^{1/3}}$ obtained by Sanders [21]. Another interesting consequence is a result related to a problem considered by Konyagin and Łaba [16]. Inserting our estimates in Sanders' proof [23], we infer that there exists $c > 0$ such that for every finite set $A \subseteq \mathbb{R}$ and every transcendental $\alpha \in \mathbb{R}$

$$|A + \alpha \cdot A| \ge (\log |A|)^{c \log \log |A|} |A|.$$

Let us also formulate our version of Bogolyubov-Ruzsa lemma for $\mathbb{F}_2^n$, in which case it reads.

Theorem 9 *There are absolute constants* C_1, C_2 *such that if* $A \subseteq \mathbb{F}_2^n$ *has density* δ*, then* $6A$ *contains a subspace* H *of codimension* $C_1 \exp(C_2\sqrt{\log(1/\delta)})$.

Other consequences of Lemma 3 in $\mathbb{F}_2^n$ the reader may find in [9]. The last application provides a progress toward Ruzsa's conjecture, which states that for every finite set of squares A we have $|A+A| \gg |A|^{2-\varepsilon}$. Currently, the best estimate $|A+A| \geq (\log|A|)^{1/12}|A|$ is due to Chang [5].

Theorem 10 *There is a positive constant* c *such that for every finite set* $A \subseteq \{1^2, 2^2, \dots\}$ *we have*

$$|A+A| \geq (\log|A|)^{c \log\log|A|}|A|.$$

Proof. Set $|A+A| = K|A|$. By Theorem 8 there is a proper generalized arithmetic progression $P = P_1 + \dots + P_d$ with $d \leq K^{C(\log K)^{-1/2}}$ and $\mathrm{size}(P) \leq 48K^{24}|A|$ such that

$$|A \cap P| \geq e^{-K^{C(\log K)^{-1/2}}}|A|.$$

Suppose that $|P_1| = \max_i |P_i| \geq |P|^{1/d}$ and write $P = \bigcup_{x \in P_2+\dots+P_d}(P_1 + x)$. Now, we make use of a theorem of Bombieri and Zannier [2] stating that an arithmetic progression of length k contains $O(k^{2/3+\varepsilon})$ squares. Therefore, we have

$$|A \cap P| \leq \sum_{x \in P_2+\dots+P_d} |(P_1+x) \cap A| \ll \sum_{x \in P_2+\dots+P_d} |P_1|^{3/4} \leq (48K^{24}|A|)^{1-1/(4d)},$$

hence

$$K \geq (\log|A|)^{c\log\log|A|},$$

and the assertion follows. □

It would be interesting to strengthen Lemma 3 since, clearly, each such improvement would result in better bounds in Freiman's theorem. The argument presented here had one major weakness – it works only for small sets X, while one wish to have an analog of this result which is valid also for sets X which are roughly of the same order as Y. Indeed, if one could prove a version of Lemma 3 with $|X| \sim |Y|$, and, say $\varepsilon = 1/20$, then Balog-Szemerédi-Gowers theorem would give $|X' + X'| \leq K^{1-c}|X'|$, for some $X' \subseteq X$ and $c > 0$. Then, one could iterate this process and, most probably, get a result very close to Freiman-Ruzsa's polynomial conjecture.

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