New bounds in Balog-Szemerédi-Gowers theorem

By

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Abstract

We prove, in particular, that every finite subset A of an abelian group with the additive energy $\kappa|A|^3$ contains a set A' such that $|A'| \gg \kappa|A|$ and $|A'-A'| \ll \kappa^{-4}|A'|$.

1 Introduction

Balog-Szemerédi-Gowers theorem [2], [4] is one of the most important tool in additive combinatorics. It asserts that very finite subset A of an abelian group with large additive energy $\mathsf{E}(A)$ i.e. with many solutions to the equation $x+y=x'+y'$, contains a large subset A' with small sumset $A'+A'$. This result has huge number of deep applications, see for example [3], [4]. The first effective bound on the size of A' and $A'+A'$ was given by Gowers and since then the theorem was improved many times and the currently best estimate is due to Balog [1]. He showed, in particular, that if $\mathsf{E}(A)=\kappa|A|^3$ then there exists $A' \subseteq A$ such that $|A'| \gg \kappa|A|$ and $|A'-A'| \ll \kappa^{-6}|A'|$.

Here we prove two theorems, which provides further improvements.

Theorem 1.1 *Let A be a subset of an abelian group such that $\mathsf{E}(A)=\kappa|A|^3$. Then there exists $A' \subseteq A$ such that $|A'| \gg \kappa|A|$ and*

$$|A'-A'| \ll \kappa^{-4}|A'|\,.$$

Theorem 1.2 *Let A be a subset of an abelian group such that $\mathsf{E}(A)=\kappa|A|^3$. Then there exist $A', B' \subseteq A$ such that $|A'|,|B'| \gg \kappa^{3/4}\log^{-5/4}(1/\kappa)|A|$ and*

$$|A'-B'| \ll \kappa^{-7/2}\log^{5/2}(1/\kappa)(|A'||B'|)^{1/2}\,.$$

Theorem 2 provides stronger estimates, however Theorem 1 allows us to take $A'=B'$. Our general strategy is essentially still the same as in [1], [2], [4], [6] i.e. we show that each element from $A'-A'$ has many representations in the form $(a_1-a_2)-(a_3-a_4)$, $a_1,a_2,a_3,a_4 \in A$. Our improvements comes from considering different candidates for A'. In previous works the authors looked for A' among dense subsets of $A\cap(P+s)$, where $P \subseteq A-A$ is the popular difference set. Here we will pick $A' \subseteq A\cap(A+s)$. Bounds in Theorem 2 follows from combining both methods.

Notation. Let A be a finite subset of an abelian group $\mathbf{G}$. We will write $A(x)$ for the indicator function of the set A. Define

$$(A*B)(x)=\sum_{y\in\mathbf{G}} A(y)B(x-y)$$

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$$(A \circ B)(x) = \sum_{y \in \mathbf{G}} A(y)B(x+y)\,,$$

so that $(A \circ B)(x)$ is the number of representations in the form $x = b - a$, $a \in A, b \in B$, while $(A * B)(x)$ is the number of representations in the form $x = b + a$, $a \in A, b \in B$. For $\beta \in \mathbb{R}$ put

$$\mathsf{E}_\beta(A) = \sum_x (A \circ A)(x)^\beta\,.$$

Note that $\mathsf{E}(A) = \mathsf{E}_2(A)$. By log we always mean $\log_2$.

2 Proof of Theorem 1.1

We start with a variant of Sanders lemma [5]. Let P_γ be the set of $\gamma-$popular differences i.e. the set of all x such that $(A \circ A)(x) \geqslant \gamma |A|$.

Lemma 2.1 *Let A be a finite subset of an abelian group $\mathbf{G}$ and let $c > 0$. Suppose that $\mathsf{E}(A) = \kappa|A|^3$. Then there exists a set $X \subseteq A$ of size at least $\frac{1}{3}\kappa|A|$ such that*

$$\sum_x (X \circ X)(x) P_{c\kappa}(x) \geqslant (1 - 16c)|X|^2\,.$$

P r o o f. Observe that

$$\sum_{(A\circ A)(x) \leqslant \frac{1}{2}\kappa|A|} (A \circ A)(x)^2 \leqslant \frac{1}{2}\kappa|A| \sum_x (A \circ A)(x) = \frac{\kappa}{2}|A|^3 = \frac{1}{2}\mathsf{E}(A) \qquad (1)$$

For $0 \leqslant i \leqslant \lceil \log(1/\kappa) \rceil$, let $Q_i = \{x : 2^{-i-1}|A| < (A \circ A)(x) \leqslant 2^{-i}|A|\}$. Hence, putting $\delta_i = \kappa^{-1}2^{-2i}$, in view of (1) we have

$$\sum_i \delta_i |Q_i| = \frac{1}{\kappa|A|^2} \sum_i \frac{|A|^2}{2^{2i}} |Q_i| \geqslant \frac{\mathsf{E}(A)}{2\kappa|A|^2} = \frac{1}{2}|A|\,.$$

Let S be the set of all pairs $(a, b) \in A^2$ such that $a - b \notin P_{c\kappa}$. Then

$$\sum_i \sum_{(a,b) \in S} |(A - a) \cap (A - b) \cap Q_i| \leqslant \sum_{(a,b)\in S} |(A - a) \cap (A - b)| \leqslant c\kappa|A||S| \leqslant c\kappa|A|^3\,.$$

Therefore, there exists i_0 such that

$$\sum_{(a,b)\in S} |(A - a) \cap (A - b) \cap Q_{i_0}| \leqslant 2c\kappa\delta_{i_0}|Q_{i_0}||A|^2\,. \qquad (2)$$

Put $Q = Q_{i_0}$, $\lambda = 2^{-i_0}$, $\delta = \delta_{i_0}$ and $N = |Q|$. We choose at random $s \in \mathbf{G}$ such that

$$\mathbb{P}(s = x) = \frac{Q(x)}{N}\,,$$

for every $x \in \mathbf{G}$. Set $X = A \cap (A + s)$ and observe that $a \in X$ if and only if $a \in A$ and $s \in a - A$, hence

$$\mathbb{P}(a \in X) = \frac{A(a)|(a - A) \cap Q|}{N} = \frac{A(a)(A * Q)(a)}{N}\,,$$

and

$$\mathbb{E}|X| = N^{-1} \sum_{a \in A} (A * Q)(a) = N^{-1} \sum_{x \in Q} (A \circ A)(x) \geqslant \frac{1}{2}\lambda|A|\,.$$

Let T be the set of all pairs $(a,b)\in X^2$ such that $a-b\notin P_{c\kappa}$. Then

$$\mathbb{P}(a,b\in X)=N^{-1}\sum_{x\in(A-a)\cap(A-b)}Q(x)=N^{-1}|(A-a)\cap(A-b)\cap Q|\,,$$

so that by (2) we have

$$\mathbb{E}|T|=\sum_{(a,b)\in S}\mathbb{P}(a,b\in X)=N^{-1}\sum_{(a,b)\in S}|(A-a)\cap(A-b)\cap Q|\leqslant 2c\kappa\delta|A|^2=2c\lambda^2|A|^2.$$

Therefore

$$\mathbb{E}\big(|X|^2-(16c)^{-1}|T|\big)\geqslant\frac{1}{8}\lambda^2|A|^2\,,$$

so there exists s such that

$$|X|^2-(16c)^{-1}|T|\geqslant\frac{1}{8}\lambda^2|A|^2.$$

In particular, $|X|\geqslant\frac{1}{2\sqrt{2}}\lambda|A|\geqslant\frac{1}{3}\kappa|A|$ and $|T|\leqslant 16c|X|^2$, which completes the proof. ■

P r o o f o f T h e o r e m 1.1. Let X be a set given by Lemma 2.1 applied for $c=1/128$, and consider the following graph

$$\mathcal{H}=\Big\{(x,y)\in X^2:(A\circ A)(x-y)\geqslant\frac{1}{128}\kappa|A|\Big\}\,.$$

By Lemma 2.1 $\mathcal{H}$ has at least $(7/8)|X|^2$ edges. Denote by A' the set of all elements $x\in X$ of degree at least $(3/4)|X|$ in $\mathcal{H}$. Then clearly $|A'|\geqslant|X|/2\gg\kappa|A|$. Take any $a,b\in A'$, then there are at least $|X|/2$ elements $y\in Y$ such that $(a,y),(b,y)\in\mathcal{H}$. Therefore

$$a-b=(a-y)-(b-y)$$

has $\gg\kappa^3|A|^3$ representations in the form $(a_1-a_2)-(a_3-a_4)$, $a_1,a_2,a_3,a_4\in A$. Thus

$$\kappa^3|A|^3|A'-A'|\ll|A|^4$$

and the assertion follows. ■

3 Proof of Theorem 1.2

We will need another version of Lemma 2.1 that make use of the E_3-energy.

Lemma 3.1 *Let A be a finite subset of an abelian group $\mathbf{G}$ and let $c>0$. Suppose that $\mathsf{E}(A)=\kappa|A|^3$ and $\mathsf{E}_3(A)=M\kappa^2|A|^4$. Then there exist $M\kappa\leqslant\lambda\leqslant 1$, and $X\subseteq A$ of size at least $\frac{1}{3}\lambda|A|$ such that*

$$\sum_x(X\circ X)(x)P_\gamma(x)\geqslant(1-16c)|X|^2\,,$$

where $\gamma=cM\kappa^2\lambda^{-1}$.

P r o o f. Note that the straightforward inequalities $(\mathsf{E}(A)/|A|)^2\leqslant\mathsf{E}_3(A)\leqslant|A|E(A)$ imply that $1\leqslant M\leqslant\kappa^{-1}$. In view of $\mathsf{E}(A)=\kappa|A|^3$, we have

$$\begin{aligned}\sum_{(A\circ A)(x)\geqslant\frac12 M\kappa|A|}(A\circ A)(x)^3&=\mathsf{E}_3(A)-\sum_{(A\circ A)(x)<\frac12 M\kappa|A|}(A\circ A)(x)^3\\&\geqslant\mathsf{E}_3(A)-\frac{1}{2}M\kappa|A|\sum_x(A\circ A)(x)^2=\frac{1}{2}\mathsf{E}_3(A)\,.\end{aligned}\tag{3}$$

For $0 \leqslant i \leqslant \lceil \log(M/\kappa) \rceil$, let $Q_i = \{x : 2^{-i-1}|A| < (A \circ A)(x) \leqslant 2^{-i}|A|\}$. Putting $\varepsilon_i = \kappa^{-2}M^{-1}2^{-3i}$, by (3) we have

$$\sum_i \varepsilon_i |Q_i| = \frac{1}{\kappa^2 M|A|^3} \sum_i \frac{|A|^3}{2^{3i}} |Q_i| \geqslant \frac{\mathsf{E}_3(A)}{2\kappa^2 M|A|^3} \geqslant \frac{1}{2}|A| .$$

Again, let S be the set of all pairs $(a, b) \in A^2$ such that $a - b \notin P_\gamma$. Using similar argument as in Lemma 2.1 we infer that there exists $\lambda \geqslant M\kappa$ such that for $Q = \{x : \frac{1}{2}\lambda|A| < (A \circ A)(x) \leqslant \lambda|A|\}$ we have

$$\sum_{(a,b)\in S} |(A - a) \cap (A - b) \cap Q| \leqslant 2\gamma\varepsilon N|A|^2 ,$$

where $\varepsilon = \kappa^{-2}M^{-1}\lambda^3$ and $N := |Q|$. Again we choose at random $s \in \mathbf{G}$ such that

$$\mathbb{P}(s = x) = \frac{Q(x)}{N} .$$

for every $x \in \mathbf{G}$. Put $X = A \cap (A + s)$ and observe that

$$\mathbb{E}|X| = N^{-1} \sum_{a\in A} (A * Q)(a) = N^{-1} \sum_{x\in Q} (A \circ A)(x) \geqslant \frac{1}{2}\lambda|A| .$$

Let T be the set of all pairs $(a, b) \in X^2$ such that $a - b \notin P_\gamma$. Then $\mathbb{E}|T| \leqslant 2\gamma\varepsilon|A|^2 = 2c\lambda^2|A|^2$, so that

$$\mathbb{E}\big(|X|^2 - (16c)^{-1}|T|\big) \geqslant \frac{1}{8}\lambda^2|A|^2 .$$

Thus, there exists s such that $|X| \geqslant \frac{1}{3}\lambda|A|$ and $|T| \leqslant 16c|X|^2$. ∎

The next lemma is Corollary 6.20 in [7].

Lemma 3.2 *Let $\mathcal{H} = (A, B, E)$ be a bipartite graph with $|E| \geqslant |A||B|/M$. Then there exist $A' \subseteq A$, $B' \subseteq B$ with $|A'| \geqslant |A|/6M$, $|B'| \geqslant |B|/6M$ such that every $a \in A'$ and $b \in B'$ is connected by at least $|A||B|/2^{12}M^4$ paths of length three.*

Proof of Theorem 1.2. Assume that and $\mathsf{E}_3(A) = M\kappa^2|A|^4$. Similarly as in the proof of Theorem 1.1 let X and $\lambda \geqslant M\kappa$ be given by Lemma 3.1 applied for $c = 1/128$, and consider the graph

$$\mathcal{H} = \Big\{(x, y) \in X^2 : (A \circ A)(x - y) \geqslant \frac{1}{16} M\kappa^2\lambda^{-1}|A|\Big\} .$$

By Lemma 3.1 $\mathcal{H}$ has at least $(7/8)|X|^2$ edges. Denote by A' the set of all elements $x \in X$ of degree at least $(3/4)|X|$ in $\mathcal{H}$. Then clearly $|A'| \geqslant |X|/2 \gg \lambda|A|$. Take any $a, b \in A'$, then there are at least $|X|/2$ elements $y \in Y$ such that $(a, y), (b, y) \in \mathcal{H}$. Therefore

$$a - b = (a - y) - (b - y)$$

has $\gg \kappa^3|A|^3$ representations in the form $(a_1 - a_2) - (a_3 - a_4)$, $a_1, a_2, a_3, a_4 \in A$. Thus

$$|A' - A'| \ll M^{-2}\kappa^{-4}|A'| . \qquad (4)$$

In the next step of the proof we obtain another estimate. Observe that

$$\sum_{(A\circ A)(x)>2M\kappa|A|} (A \circ A)(x)^2 \leqslant \frac{\mathsf{E}_3(A)}{2M\kappa|A|} = \frac{1}{2}\mathsf{E}(A) .$$

Therefore there exists $\kappa/2 \leqslant \mu \leqslant 2M\kappa$ such that for $Q = \{x : \mu|A| < (A \circ A)(x) \leqslant 2\mu|A|\}$ we have

$$\sum_{x \in Q} (A \circ A)(x) \gg \frac{\kappa}{\mu \log M}|A|^2 . \tag{5}$$

Let $\mathcal{G} = (A, A, E)$ (here we can assume that the vertex set of $\mathcal{G}$ consists of two disjoint copies of A, for details see [7]) be a bipartite graph such that $E = \big\{\{a, b\} : a, b \in A,\ a - b \in Q\big\}$, so that by (5) $|E| = \alpha|A|^2 \gg \kappa\mu^{-1}(\log M)^{-1}|A|^2$. Therefore, by Lemma 3.2 there are sets $A', B' \subseteq A$ with $|A'|, |B'| \gg \alpha|A|$ such that every $a \in A'$ and $b \in B'$ is connected by $\gg \alpha^4|A||B|$ paths of length three in $\mathcal{G}$. Therefore, for each $a \in A'$ and $b \in B'$ there are $\gg \alpha^4|A||B|$ elements $x, y \in A$ such that $\{a, y\}, \{x, y\}, \{x, b\} \in \mathcal{G}$. Thus

$$a - b = (a - y) - (x - y) + (x - b)$$

has $\gg \mu^3|A|^3\alpha^4|A|^2$ representations in the form $(a_1 - a_2) - (a_3 - a_4) + (a_5 - a_6)$, $a_1, a_2, a_3, a_4, a_5, a_6 \in A$. Hence

$$|A' - B'| \ll \mu^{-3}\alpha^{-4}|A| \ll \mu^{-3}\alpha^{-5}(|A'||B'|)^{1/2} \leqslant \kappa^{-3}M^2 \log^5 M (|A'||B'|)^{1/2} . \tag{6}$$

We use (4) if $\kappa^{-1/4}\log^{-5/4}(1/\kappa) \leqslant M \leqslant \kappa^{-1}$, while we use (6) if $1 \leqslant M \leqslant \kappa^{-1/4}\log^{-5/4}(1/\kappa)$. In the former case we have

$$|A'| = |B'| \gg M\kappa|A| \geqslant \frac{\kappa^{3/4}}{\log^{5/4}(1/\kappa)}|A|$$

and in the latest one

$$|A'|, |B'| \gg \frac{1}{M \log M}|A| \gg \frac{\kappa^{1/4}}{\log^{1/4}(1/\kappa)}|A|,$$

which completes the proof. ■

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