Additive combinatorics

Winter semester 2016/2017

Series IV

4.1. Show that for every integer n one has $\|\alpha n\| = \|-\alpha n\| \leqslant |n|\|\alpha\|$.

4.2. Let $\alpha \in \mathbb{R}$ and let $Q \in \mathbb{N}$. Show that there exist $p \in \mathbb{Z}$, $q \in \mathbb{N}$ such that $(p, q) = 1$, $q \leqslant Q$ and $|\alpha - p/q| \leqslant (qQ)^{-1}$.

Hint: Apply the Pigeonhole Principle.

4.3. Suppose that $\alpha_1, \ldots, \alpha_n \in \mathbb{R}$ and $t \in \mathbb{N}$. Prove that there exist $q \in \mathbb{N}$ such that $q \leqslant t^n$ and $\|q\alpha_i\| < 1/t$.

Wskazówka: Apply the Pigeonhole Principle.

4.4. Let $t_1, \ldots, t_k \in \mathbb{N}$. Show that there exists $x \in \mathbb{R}$ such that $\|xt_i\| \geqslant 1/(2k)$, for each $1 \leqslant i \leqslant k$.

4.5.* Suppose that $\{1, 2, \ldots, 2^k\} \subseteq \mathrm{Span}(A) = \{\sum_{a \in A} \varepsilon_a a : \varepsilon_a = 0, \pm 1\}$. Show that $|A| \gg k/\log k$.

4.6. Prove a lower bound for the size of a set A satisfying $\{1!, 2!, \ldots, k!\} \subseteq \mathrm{Span}(A)$.

4.7.* Show that if $A \subseteq \mathbb{Z}_p$ and $|A + A| < \frac{1}{100}(\log p)^2$, then there is $x \in \mathbb{Z}_p^*$ such that

$$\|xa/p\| \leqslant 1/4, \quad \text{for each } \ a \in A.$$

4.8. Let X be a $k-$element set, for a fix k. Provide bounds for the size of a set $A \subseteq [N]$ such that:

a) $(A - A) \cap X = \emptyset$,

b*) $(A + A) \cap X = \emptyset$.