Additive combinatorics

Winter semester 2016/2017

Series VIII

8.1. Let $\Gamma \subseteq \mathbb{F}_p^*$ be a multiplicative subgroup. Show that

$$\max_{r \neq 0} |\widehat{\Gamma}(r)| \leqslant \sqrt{p}.$$

8.2. Does there exists a constant $\gamma > 0$ independent on p such that $|\widehat{A}(r)| > \gamma$ for every $A \subseteq \mathbb{Z}_p$ and $r \in \mathbb{Z}_p$?

8.3. Let $A \subseteq \mathbb{Z}_N$, $|A| = \delta N$. Prove that if $\mathsf{E}(A) = \delta|A|^3 + \varepsilon N^3$, then $\max_{r\neq 0} |\widehat{A}| \leqslant \varepsilon^{1/4} N$.

8.4. Let $A, B \subseteq \mathbb{Z}_N$. Suppose that $|\widehat{A}(r)| \geqslant \alpha|A|$ and $|\widehat{B}(r)| \geqslant \beta|B|$ for every $r \in \Delta$. Prove that there exists $x \in \mathbb{Z}_N$ such that

$$(A * B)(x) \geqslant (1 + \alpha^2\beta^2|\Delta|)\frac{|A||B|}{N}.$$

8.5. Let $A \subseteq [N]$ and put $A(r, q) = \{a \in A : a \equiv r \pmod{q}\}$. Suppose that for every $r \in [q]$ we have $|f(r/q)| = |\sum_{a\in A} e^{-2\pi i r a/q}| > (1 - \varepsilon)|A|$. Show that for some $r_0 \in [q]$ we have $|A(r_0, q)| > (1 - 2\varepsilon)|A|$.

8.6. Let $f(\alpha) = \sum_{a\in A} e^{2\pi i a\alpha}$, where $A \subseteq [N]$. Show that

$$|f(\alpha) - f(\beta)| = O(N|A||\alpha - \beta|).$$